COISOTROPIC AND POLAR ACTIONS ON COMPACT IRREDUCIBLE HERMITIAN SYMMETRIC SPACES

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Abstract. We obtain the full classification of coisotropic and polar isometric actions of compact Lie groups on irreducible Hermitian symmetric spaces.

1. Introduction

The aim of the present paper is to investigate polar and coisotropic actions on compact irreducible Hermitian symmetric spaces.

The action of a compact Lie group K of isometries on a Riemannian manifold (M, g) is called *polar* if there exists a properly embedded submanifold Σ which meets every $K-$orbit and is orthogonal to the $K-$orbits in all common points. Such a submanifold Σ is called a *section* (see [21], [22]) and it is automatically totally geodesic; if it is flat, the action is called *hyperpolar*.

Let (M, g) be a compact Kähler manifold with Kähler form ω and let K be a compact connected Lie subgroup of its full isometry group. The K-action is called *coisotropic* or *multiplicity free* if the principal K-orbits are coisotropic with respect to ω [15]. Notice that the existence of an open subset consisting of coisotropic orbits implies that all $K-$orbits are coisotropic, see [15]. Multiplicity free representations form a very restricted class of representation. Nevertheless they are very important since every "nice" result in the invariant theory of particular representations can be traced back to a multiplicity free representation. This holds for example for a Capelli identities [14] and also all of Weyl's first and second fundamental theorems can be explained by some multiplicity freeness result.

Kac [16] and Benson and Ratcliff [2] have given the classification of linear multiplicity free representations, from which one has the full classification of coisotropic actions on $Gr(k, n)$ for $k = 1$, i.e. on the complex projective space. In a recent paper [3] the complete classification of polar and coisotropic actions on complex Grassmannians has been obtained while in [24], as an application of the main result, it was given the complete classification of this kind of actions on the quadric $\mathrm{SO}(n+2)/\mathrm{SO}(2) \times \mathrm{SO}(n)$. Hence it is natural to investigate coisotropic and polar actions on the other compact irreducible Hermitian symmetric spaces, which are $\mathrm{SO}(2m)/\mathrm{U}(m)$, $\mathrm{Sp}(m)/\mathrm{U}(m)$, $\mathrm{E}_7/\mathrm{T}^1 \cdot \mathrm{E}_6$ and $\mathrm{E}_6/\mathrm{T}^1 \cdot \mathrm{Spin}(10)$. Our main result is given in the following

Theorem 1.1. *Let K be a compact connected Lie subgroup of* $\mathrm{Sp}(m)$, *respectively* $\mathrm{SO}(2m)$, *acting non-transitively on the Hermitian symmetric space* $M = \mathrm{Sp}(2m)/\mathrm{U}(m)$, *respectively* $M = \mathrm{SO}(2m)/\mathrm{U}(m)$. *Then K acts coisotropically on M if and only if its Lie algebra $\mathfrak{k}$, up to conjugation in* $\mathfrak{sp}(2m)$, *respectively* $\mathfrak{o}(2m)$,

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contains one of the Lie algebras appearing in Table 1. In Table 2 we list, up to conjugation, all the subgroups of $\mathrm{E}_7,\ \mathrm{E}_6$, *which act non-transitively and coisotropically on* $\mathrm{E}_7/\mathrm{T}^1\cdot\mathrm{E}_6$ *and* $\mathrm{E}_6/\mathrm{T}^1\cdot\mathrm{Spin}(10)$ *respectively.*

TABLE 1.

$\mathfrak{k}$	M	conditions
$\mathfrak{u}(1)$	$\mathrm{Sp}(1)/\mathrm{U}(1)$	
$\mathfrak{su}(m)$	$\mathrm{Sp}(m)/\mathrm{U}(m)$	$m\geq 2$
$\mathfrak{sp}(k)+\mathfrak{sp}(m-k)$	$\mathrm{Sp}(m)/\mathrm{U}(m)$	$1\leq k\leq m-1$
$\mathfrak{sp}(m-1)+\mathfrak{u}(1)$	$\mathrm{Sp}(m)/\mathrm{U}(m)$	$m\geq 2$
$\mathfrak{sp}(m)+\mathfrak{sp}(1)+\mathfrak{sp}(1)$	$\mathrm{Sp}(m+2)/\mathrm{U}(m+2)$	
$\mathbb{R}(0)$	$\mathrm{SO}(4)/\mathrm{U}(2)$	$\mathbb{R}(0)$ line in $\mathfrak{t}_2\times\mathfrak{z}$
$\mathfrak{z}+\mathfrak{t}_3$	$\mathrm{SO}(6)/\mathrm{U}(3)$	
$\mathbb{R}(\frac{1}{2k})+\mathfrak{su}(2k)$	$\mathrm{SO}(4k+2)/\mathrm{U}(2k+1)$	$k\geq 2,\ \mathbb{R}(\frac{1}{2k})$ line in $\mathfrak{a}\times\mathfrak{z}$
$\mathbb{R}+\mathfrak{su}(2k+1)$	$\mathrm{SO}(4k+4)/\mathrm{U}(2k+2)$	$k\geq 2,\ \mathbb{R}$ means any line in $\mathfrak{a}\times\mathfrak{z}$
$\mathbb{R}(0)+\mathfrak{su}(3)$	$\mathrm{SO}(8)/\mathrm{U}(4)$	$\mathbb{R}(0)$ line in $\mathfrak{a}\times\mathfrak{z}$
$\mathfrak{z}+\mathfrak{su}(2)$	$\mathrm{SO}(6)/\mathrm{U}(3)$	
$\mathfrak{su}(m)$	$\mathrm{SO}(m)/\mathrm{U}(m)$	$m\geq 2$
$\mathfrak{z}+\mathfrak{sp}(2)$	$\mathrm{SO}(8)/\mathrm{U}(4)$	
$\mathfrak{sp}(1)+\mathfrak{sp}(2)$	$\mathrm{SO}(8)/\mathrm{U}(4)$	$\mathfrak{sp}(1)\otimes\mathfrak{sp}(2)\subseteq\mathfrak{so}(8)$
$\mathfrak{so}(k)+\mathfrak{so}(2m-k)$	$\mathrm{SO}(2m)/\mathrm{U}(m)$	
$\mathfrak{so}(2m-2)$	$\mathrm{SO}(2m)/\mathrm{U}(m)$	$m\geq 3$
$\mathfrak{so}(2m-6)+\mathfrak{u}(3)$	$\mathrm{SO}(2m)/\mathrm{U}(m)$	$m\geq 5$
$\mathfrak{so}(2m-4)+\mathfrak{u}(2)$	$\mathrm{SO}(2m)/\mathrm{U}(m)$	$m\geq 4$
$\mathfrak{so}(2m)+\mathbb{R}(1,-1)$	$\mathrm{SO}(2(m+2))/\mathrm{U}(m+2)$	$m\geq 5$, $\mathbb{R}(1,-1)\subseteq\mathfrak{so}(2)\times\mathfrak{so}(2)\subseteq\mathfrak{so}(4)$
$\mathfrak{so}(4)+\mathfrak{so}(2)+\mathfrak{so}(2)$	$\mathrm{SO}(8)/\mathrm{U}(4)$	
$\mathfrak{g}_2$	$\mathrm{SO}(8)/\mathrm{U}(4)$	$\mathfrak{g}_2\subset\mathfrak{so}(7)\subset\mathfrak{so}(8)$

TABLE 2.

	$M=\mathrm{E}_7/\mathrm{T}^1\cdot\mathrm{E}_6$			
maximal subgroups	$\mathrm{T}^1\cdot\mathrm{E}_6$	$\mathrm{SU}(2)\cdot\mathrm{Spin}(12)$	$\mathrm{SU}(8)/\mathbb{Z}_2$	
		$\mathrm{T}^1\cdot\mathrm{Spin}(12)$	$\mathrm{S}(\mathrm{U}_1\times\mathrm{U}_7)/\mathbb{Z}_2$	
		$\mathrm{SU}(2)\cdot\mathrm{Spin}(11)$	$\mathrm{SU}(7)/\mathbb{Z}_2$	
	$M=\mathrm{E}_6/\mathrm{T}^1\cdot\mathrm{Spin}(10)$			
maximal subgroups	$\mathrm{T}^1\cdot\mathrm{Spin}(10)$	$\mathrm{Sp}(1)\cdot\mathrm{SU}(6)$	$\mathrm{Sp}(4)/\mathbb{Z}_2$	F_4
	$\mathrm{Spin}(10)$	$\mathrm{T}^1\cdot\mathrm{SU}(6)$		
	$\mathrm{T}^1\cdot\mathrm{Spin}(9)$	$\mathrm{Sp}(1)\cdot\mathrm{U}(5)$		
	$\mathrm{T}^1\cdot(\mathrm{T}^1\times\mathrm{Spin}(8))$	$\mathrm{T}^1\cdot\mathrm{U}(5)$		

All the Lie algebras listed in the first column, unless explicitly specified, are meant to be standardly embedded into $\mathfrak{sp}(m)$, respectively $\mathfrak{so}(2m)$, e.g. $\mathfrak{sp}(m)+\mathfrak{u}(1)\subset\mathfrak{sp}(m)+\mathfrak{sp}(2)\subset\mathfrak{sp}(m)$, $\mathfrak{so}(2m-3)+\mathfrak{u}(3)\subset\mathfrak{so}(2m-3)+\mathfrak{so}(6)\subset\mathfrak{so}(2m)$. The notations used in Table 1 are as follows. We denote with $\mathfrak{z}$ the one dimensional center of $Lie(\mathrm{U}(m))$, with $\mathfrak{a}$ the centralizer of the semisimple part of $\mathfrak{k}$ in $\mathfrak{su}(m)\subset Lie(\mathrm{U}(m))$ and with $\mathfrak{t}_m$ the subalgebra of a maximal torus of $\mathrm{SU}(m)\subset\mathrm{U}(m)$ With this notation $\mathbb{R}(\alpha)$ denotes any line in $\mathfrak{a}\times\mathfrak{z}$ different from $y=\alpha x$ while

$\mathbb{R}(1,-1) \subseteq \mathfrak{so}(2)+\mathfrak{so}(2) \subseteq \mathfrak{so}(4)$ means any line in the plane $\mathfrak{so}(2) \times \mathfrak{so}(2)$ different from $y = x$ and $y = -x$. Finally, in Table 2 the juxtaposition $A \cdot B$ of two groups generally denotes the quotient $A \times_{\mathbb{Z}_2} B$.

Victor Kac [16] obtained a complete classification (Tables Ia, Ib, in the Appendix) of irreducible multiplicity free actions (σ, V). Most of these include a copy of the scalars $\mathbb{C}$ acting on V. We will say that a multiplicity free action (σ, V) of a complex group G is *decomposable* if we can write V as the direct sum $V = V_1 \oplus V_2$ of proper $\sigma(G)$-invariant subspaces in such a way that $\sigma(G) = \sigma_1(G) \times \sigma_2(G)$, where σ_i denotes the restriction of σ to V_i. If V does not admit such a decomposition then we say that (σ, V) is an *indecomposable* multiplicity free action. C. Benson and G. Ratcliff have given the complete classification of indecomposable multiplicity free actions (Tables IIa, IIb in the Appendix). We recall here their theorem (Theorem 2, page 154 [2])

Theorem 1.2. *Let (σ, V) be a regular representation of a connected semisimple complex algebraic group G and decompose V as a direct sum of $\sigma(G)$-irreducible subspaces, $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$. The action of $(\mathbb{C}^*)^r \times G$ on V is an indecomposable multiplicity free action if and only if either*

- **(1):** *$r = 1$ and $\sigma(G) \subseteq \mathrm{G}\ell(V)$ appears in Table Ia (see the Appendix);*
- **(2):** *$r = 2$ and $\sigma(G) \subseteq \mathrm{G}\ell(V_1) \times \mathrm{G}\ell(V_2)$ appears in Tables IIa and IIb (see the Appendix).*

In [2] are also given conditions under which one can *remove* or *reduce* the copies of the scalars preserving the multiplicity free action. Obviously if an action is coisotropic it continues to be coisotropic also when this action includes another copy of the scalars. We will call *minimal* those coisotropic actions in which the scalars, if they appear, cannot be reduced.

Let K be a compact group acting isometrically on a compact Kähler manifold M. This action is automatically holomorphic by a theorem of Kostant (see [17], vol I, page 247) and it induces by compactness of M an action of the complexified group $K^{\mathbb{C}}$ on M. We say that M is $K^{\mathbb{C}}$-*almost homogeneous* if $K^{\mathbb{C}}$ has an open orbit in M. If all Borel subgroups of $K^{\mathbb{C}}$ act with an open orbit on M, then the $K^{\mathbb{C}}$-open orbit Ω is called a *spherical homogeneous space* and M is called a *spherical embedding* of Ω. We will briefly recall some results that will be used in the sequel.

Theorem 1.3. [15] *Let M be a connected compact Kähler manifold with an isometric action of a connected compact group K that is also Poisson. Then the following conditions are equivalent:*

- (i) *The K-action is coisotropic.*
- (ii) *The cohomogeneity of the K action is equal to the difference between the rank of K and the rank of a regular isotropy subgroup of K.*
- (iii) *The moment map $\mu : M \to \mathfrak{k}^*$ separates orbits.*
- (iv) *The Kähler manifold M is projective algebraic, $K^{\mathbb{C}}$-almost homogeneous and a spherical embedding of the open $K^{\mathbb{C}}$-orbit.*

We remark here that conditions (i) to (iii) are equivalent even without the hypothesis of compactness on M (see [15]).

As an immediate consequence of the above theorem one can deduce, under the same hypotheses on K and M, two simple facts that will be frequently used in our classification:

1 Let p be a fixed point on M for the K-action, or Kp a complex K-orbit, then the K-action is coisotropic if and only if the slice representation is coisotropic (see [15] page 274).
2 *dimensional condition.* If K acts coisotropically on M the dimension of a Borel subgroup B of $K^{\mathbb{C}}$ is not less than the dimension of M.

A relatively large class of coisotropic actions is provided by polar ones. A result due to Hermann ([13]) states that given K a compact Lie group and two symmetric subgroups H_1,$H_2 \subseteq K$, then H_i acts hyperpolarly on K/H_j for $i, j \in 1, 2$. This kinds of action are coisotropic since for [24] a polar action on an irreducible compact homogeneous Kähler manifold is coisotropic.

Once we shall determined the complete list of coisotropic actions on compact irreducible Hermitian symmetric spaces we have also investigated which ones are polar. Dadok [7], Heintze and Eschenburg [13] have classified the irreducible polar linear representations, while I.Bergmann [4] has found all the reducible ones. Using their results we determine in section 7 the complete classification of the polar actions on the following Hermitian symmetric spaces $\mathrm{SO}(2m)/\mathrm{U}(m)$, $\mathrm{Sp}(m)/\mathrm{U}(m)$, $\mathrm{E}_6/\mathrm{T}^1\cdot \mathrm{Spin}(10)$, $\mathrm{E}_7/\mathrm{T}^1\cdot\mathrm{E}_6$. An interesting consequence of this classification is that the polar actions on these manifolds are just the hyperpolar ones. The same result holds on the quadrics (see [24]) and on the complex Grassmannians (see [3]). In particular, we have the following

Proposition 1.1. *A polar action on a compact irreducible Hermitian symmetric space of rank bigger than one is hyperpolar.*

This is in contrast to complex projective space or more generally to rank one symmetric spaces that admit many polar actions that are not hyperpolar (see [23]).

We point out also that on the Hermitian symmetric space $M = \mathrm{E}_7/\mathrm{T}^1\cdot\mathrm{E}_6$, respectively $M = \mathrm{Sp}(m)/\mathrm{U}(m)$, our result implies that a compact connected Lie subgroup K of E_7, respectively $\mathrm{Sp}(m)$, acts polarly on M if and only if K is a symmetric group.

We mention the following conjecture concerning the nature of polar actions on compact symmetric spaces.

Conjecture 1. *A polar action on a compact symmetric space of rank bigger than one is hyperpolar.*

In particular in Proposition 1.1 is given the positive answer in the class of compact irreducible Hermitian symmetric spaces.

The complete classification of polar actions on the compact irreducible Hermitian symmetric spaces, which they have been investigated in this paper, is given in the following

Theorem 1.4. *Let K be a compact connected Lie subgroup of* $\mathrm{SO}(2m)$, *respectively* $\mathrm{Sp}(m)$, *acting non-transitively on* $M = \mathrm{SO}(2m)/\mathrm{U}(m)$ *respectively* $M = \mathrm{Sp}(m)/\mathrm{U}(m)$. *Then K acts polarly on M if and only if its Lie algebra $\mathfrak{k}$ is conjugate, in $\mathfrak{o}(2m)$, respectively $\mathfrak{sp}(m)$, to one of the Lie algebras appearing in Table* 3. *In Table* 4 *we list, up to conjugation, all the subgroups of* E_7, E_6, *which act non-transitively and polarly on* $\mathrm{E}_7/\mathrm{T}^1\cdot\mathrm{E}_6$ *and* $\mathrm{E}_6/\mathrm{T}^1\cdot\mathrm{Spin}(10)$ *respectively. In particular on these manifolds is that polar actions are hyperpolar.*

Table 3.

$\mathfrak{k}$	M	conditions
$\mathfrak{u}(m)$	$\mathrm{Sp}(m)/\mathrm{U}(m)$	$m \geq 1$
$\mathfrak{sp}(k)+\mathfrak{sp}(m-k)$	$\mathrm{Sp}(2m)/\mathrm{U}(m)$	
$\mathfrak{u}(m)$	$\mathrm{SO}(2m)/\mathrm{U}(m)$	
$\mathfrak{su}(m)$	$\mathrm{SO}(2m)/\mathrm{U}(m)$	m odd
$\mathfrak{so}(k)+\mathfrak{so}(2m-k)$	$\mathrm{SO}(2m)/\mathrm{U}(m)$	
$\mathfrak{so}(2m-2)$	$\mathrm{SO}(2m)/\mathrm{U}(m)$	$m \geq 3$
$\mathfrak{g}_2$	$\mathrm{SO}(8)/\mathrm{U}(8)$	$\mathfrak{g}_2 \subset \mathfrak{so}(7) \subset \mathfrak{so}(8)$
$\mathbb{R}(0)$	$\mathrm{SO}(4)/\mathrm{U}(2)$	

Table 4.

	$M = \mathrm{E}_7/\mathrm{T}^1 \cdot \mathrm{E}_6$		
$\mathrm{T}^1 \cdot \mathrm{E}_6$	$\mathrm{Spin}(12) \cdot \mathrm{SU}(2)$	$\mathrm{SU}(8)/\mathbb{Z}_2$	
	$M = \mathrm{E}_6/\mathrm{T}^1 \cdot \mathrm{Spin}(10)$		
$\mathrm{T}^1 \cdot \mathrm{Spin}(10)$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$\mathrm{Sp}(4)/\mathbb{Z}_2$	F_4
$\mathrm{Spin}(10)$			

We here briefly explain our method in order to prove our main theorem. Thanks to Theorem 1.3, (iv) we have that if K is a subgroup of a compact Lie group L such that K acts coisotropically on M so does L. As a consequence, in order to classify coisotropic actions on $\mathrm{SO}(2m)/\mathrm{U}(m)$ ($\mathrm{Sp}(m)/\mathrm{U}(m)$, $\mathrm{E}_7/\mathrm{T}^1 \cdot \mathrm{E}_6$, $\mathrm{E}_6/\mathrm{T}^1 \cdot \mathrm{Spin}(10)$), one may suggest a sort of "telescopic" procedure by restricting to maximal subgroups K of $\mathrm{SO}(n)$,($\mathrm{Sp}(m)$, E_7 E_6) hence passing to maximal subgroups that give rise to coisotropic actions and so on.

This paper is organized as follows. In section 2 we prove a useful result that we shall use throughout this paper. From section 3 to section 6 we give the proof of Theorem 1.1. We have divided every section in subsections in each of which we analyze separately one of the maximal subgroups of $\mathrm{SO}(m)$ respectively $\mathrm{Sp}(2m)$, E_7 and E_6. In the seventh section we give the proof of Theorem 1.4.

We enclose, in the Appendix, the tables of irreducible and reducible linear multiplicity free representations (Tables Ia, Ib and Tables IIa, IIb respectively) and the tables of maximal subgroups of $\mathrm{Sp}(2m)$, $\mathrm{SO}(n)$ and $\mathrm{SU}(n)$ (Tables **III**, **IV**,**V**).

2. Preliminaries

Let $\mathfrak{g}$ be a Lie semisimple complex algebra. We will denote by $\mathfrak{b}$ a Borel Lie algebra of $\mathfrak{g}$, whose dimension is $\frac{1}{2}(\dim \mathfrak{g} + r(\mathfrak{g}))$, where $r(\mathfrak{g})$ is the dimension of a Cartan subalgebra, namely the rank of $\mathfrak{g}$. Throughout this paper we will identify the fundamental dominant weights Λ_l with the corresponding irreducible representations. It is well known that any irreducible representation corresponds to a highest weight σ and any highest weight is of the form $\sigma = \sum_i m_i \Lambda_i$, where m_i are non-negative integers. We will denote by $d(\sigma)$ the representation degree of σ, i.e. the dimension of the vector space on which $\mathfrak{g}$ acts with the irreducible representation σ. Using the Weyl's dimensional formula it easy to check that if $m_i \geq n_i$ then $d(\sum_i m_i \Lambda_i) \geq d(\sum_i n_i \Lambda_i)$ and the equality hold if and only if $m_i = n_i$.

Lemma 2.1. *Let $\mathfrak{g}$ be a simple complex Lie algebra and let $\sigma : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on V with $d = \dim V$. Let $\mathfrak{b}$ be the Lie algebra of a Borel subgroup of $\mathfrak{g}$. Then we have*

(1) $1 + \dim \mathfrak{b} < \frac{1}{2}d(d-1)$ *except when* $\mathfrak{g} = \mathfrak{sl}(m)$ *and either* $\sigma = \Lambda_1$ *or* $\sigma = \Lambda_{m-1}$, $\mathfrak{g} = \mathfrak{sl}(2)$ *and* $\sigma = 2\Lambda_1$, $\mathfrak{g} = \mathfrak{so}(5)$ *and* $\sigma = \Lambda_2$ *(spin-representation) and* $\mathfrak{g} = \mathfrak{so}(6)$ *and either* $\sigma = \Lambda_3$ *or* $\sigma = \Lambda_2$ *(spin-representations);*

(2) $1 + \dim \mathfrak{b} < \frac{1}{2}d(d+1)$ *except when* $\mathfrak{g} = \mathfrak{sl}(m)$ *and either* $\sigma = \Lambda_1$ *or* $\sigma = \Lambda_{m-1}$;

Proof. Since the second affirmation can be deduced easily from the first, we shall prove only our first statement. Our basic references are [25] and [18] Appendix B.

Assume $\mathfrak{g} = \mathfrak{sl}(m)$. Then the dimension of the Borel subalgebra is $\dim \mathfrak{b} = \frac{1}{2}(m-1)(m+2)$. The cases $m = 2, 3$ are easy to check. If $m \geq 4$, we have $d(\Lambda_1 + \Lambda_{m-1}) \geq m + \frac{3}{2}$, $d(2\Lambda_1) \geq m + \frac{3}{2}$ and $d(\Lambda_2) \geq m + \frac{3}{2}$. In particular, for every representation $\sigma \neq \Lambda_1, \Lambda_{m-1}$, one may verify that $1+\dim \mathfrak{b} < \frac{1}{4}(2m+3)(2m+1) \leq \frac{1}{2}d(\sigma)(d(\sigma)-1)$. Assume $\mathfrak{g} = \mathfrak{sp}(m), m \geq 3$. Since $d(\sigma) > 4m > d(\Lambda_1) = 2m$, when $\sigma \neq \Lambda_1$, we have $1+\dim \mathfrak{b} = 1+m^2+m < \frac{1}{2}d(\sigma)(d(\sigma)-1)$, since $1+m^2+m < m(2m-1)$ for $m \geq 3$. If $\mathfrak{g} = \mathfrak{so}(2m+1)$, we distinguish the case $m \geq 4$ and $m = 2, 3$. When $m \geq 4$, since $d(\sigma) \geq 2m-1$, we have $1 + \dim \mathfrak{b} = m^2 + m < \frac{1}{2}d(\sigma)(d(\sigma)-1)$. If $m = 3$, since $d(\sigma) \geq 7$, one may prove that $\frac{1}{2}d(\sigma)(d(\sigma)-1) > 13$ is verified for every σ, while in the case $m = 2$ we have that $\sigma = \Lambda_2$ does not satisfy the above inequality. The case $\mathfrak{g} = \mathfrak{so}(2m)$, can be resolved as before. Indeed, if $m \geq 4$ then it is to check that $d(\sigma) \geq 2m-1$ for every σ. In particular $1 + \dim b = 1 + m^2 < (2m-1)(m-1) \leq \frac{1}{2}d(\sigma)(d(\sigma)-1)$. If $m = 3$, since $d(2\Lambda_1) = d(\Lambda_1 + \Lambda_2) = 20$, $d(\Lambda_1 + \Lambda_3) = 15$, and $d(2\Lambda_2) = d(2\Lambda_3) = 10$, one may prove that $10 < \frac{1}{2}d(\sigma)(d(\sigma)-1)$ except for $\sigma = \Lambda_i,\ i = 2, 3$. If $\mathfrak{g}$ is of type $\mathfrak{g}_2$ ($\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$) it is well known that minimal representation degree is 7 (respectively 26, 27, 56, 248) and the dimension of a Borel subalgebra is 8 (respectively 31, 42, 70, 127), then for any representation σ we have $1 + \dim \mathfrak{b} < \frac{1}{2}d(\sigma)(d(\sigma)-1)$. □

3. $M = \mathrm{Sp}(m)/\mathrm{U}(m)$

3.1. **The case $K = \rho(H)$, H simple such that $\rho \in Irr_{\mathbb{H}}(H)$, $d(\rho) = 2m$.** We briefly explain our notation, that we will use throughout this paper. Let H be a simple group. By $Irr_{\mathbb{R}}(H)$, $Irr_{\mathbb{C}}(H)$, $Irr_{\mathbb{H}}(H)$ we denote the irreducible representation of H of real, complex and quaternionic type, see [5], Chapter II, §6 .

By Table III in the Appendix, if K is the image of an irreducible quaternionic representation ρ of a simple Lie group H, i.e. $K = \rho(H)$ where $\rho \in Irr_{\mathbb{H}}(H)$, then K is a maximal group. In this section we analyze this case.

Let H be a simple group. It is well known that if $\mathfrak{h}_o$ is a simple real algebra whose complexification $\mathfrak{h}$ is simple, its irreducible representations are the restrictions of (uniquely determined) irreducible representation of $\mathfrak{h}$. Our idea is very simple: we impose the *dimensional condition.* By lemma 2.1 we have only to consider $(\mathfrak{sl}(2), \Lambda_1)$, which corresponds to $\mathrm{SU}(2) \subseteq \mathrm{U}(2) \subseteq \mathrm{SO}(4)$. This case will be studied in next section, since $\mathrm{SU}(2)$ has a fixed point.

3.2. **The fixed point case $K = \mathrm{U}(m)$.** $\mathrm{U}(m)$ has a fixed point and the slice is given by $\mathrm{S}^2(\mathbb{C}^m)$. By Tables Ia and Ib, the action is multiplicity free and the scalar can be removed when $m \geq 2$. We will now go through the maximal subgroups of $\mathrm{U}(m)$. Let $L \subset \mathrm{U}(m)$ be such that Lie(L)= $\mathfrak{z} + \mathfrak{l}_1$, where $\mathfrak{l}_1$ is a maximal subalgebra

of $\mathfrak{su}(m)$ (see Table **V** in the Appendix). By lemma 2.1 the dimensional condition is not satisfies for (i), (ii) and (v) of Table **V**. The same holds for $\mathfrak{l}_1 = \mathfrak{su}(p)+\mathfrak{su}(q)$. Indeed, the dimension of a Borel subalgebra of $(\mathfrak{z}+\mathfrak{l}_1)^{\mathbb{C}}$ is $1+\frac{1}{2}((p-1)(p+2)+(q-1)(q+2))$. The inequality $1+\frac{1}{2}((p-1)(p+2)+(q-1)(q+2)) < \frac{1}{2}(pq(pq+1))$ is always satisfies, so the action fails to be multiplicity free. Indeed, let $f(x) = x^2(q^2-1)+x(q-1)-q^2-q+2$. Then $f'(x) = 2x(q^2-1)+q-1 > 0$, for $x \geq 3$ and $f(3) = 9(q^2-1)+3(q-1)-q^2-q+2 > 0$, since $q \geq 2$. Finally, if $\mathfrak{l}_1 = \mathbb{R}+\mathfrak{su}(k)+\mathfrak{su}(m-k)$ then the slice becomes $\mathrm{S}^2(\mathbb{C}^k)\oplus(\mathbb{C}^k\otimes\mathbb{C}^{m-k})^*\oplus\mathrm{S}^2(\mathbb{C}^{m-k})$. Hence, by Tables IIa and IIb we have $k = m-k = 1$ which implies $\dim\mathfrak{l} = 2 < \dim\mathrm{S}^2(\mathbb{C}^2)$. Summing up we have the following minimal subalgebra: $\mathfrak{u}(1)$ acting on $\mathrm{Sp}(1)/\mathrm{U}(1)$ and $\mathfrak{su}(m)$ acting on $\mathrm{Sp}(m)/\mathrm{U}(m)$.

3.3. **The case** $K = \mathrm{SO}(p)\otimes\mathrm{Sp}(q),\ pq = m,\ p\geq 3,\ q\geq 1$**.** The dimension of a Borel subgroup of $K^{\mathbb{C}}$ is equal or less than $\frac{p^2}{4}+q^2+q$, while $\dim M = \frac{1}{2}pq(pq+1)$, since $m = pq$. Now, let $f(x) = x^2(2q^2-1)+2xq-4q^2-4q$. Then $f'(x) > 0$ for $x \geq 0$ and $f(3) > 0$ since $q \geq 1$. Then the $K-$action cannot be coisotropic.

3.4. **The case** $K = \mathrm{Sp}(k)\times\mathrm{Sp}(m-k)$**.** Since K is a symmetric subgroup of $\mathrm{Sp}(m)$, the $K-$action is hyperpolar. We shall analyze the subgroups of K. The manifold M parametrizes the space of Lagrangian subspaces of $\mathbb{C}^{2m}$ respect to a symplectic form. We consider $\omega(X,Y) = X^tJY$ where

$$J = \left(\begin{array}{c|c|c|c} 0 & -\mathrm{I}_k & 0 & 0 \\ \hline \mathrm{I}_k & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\mathrm{I}_{m-k} \\ \hline 0 & 0 & \mathrm{I}_{m-k} & 0 \end{array}\right) = \left(\begin{array}{c|c} J_k & 0 \\ \hline 0 & J_{m-k} \end{array}\right)$$

Let $W_o =< e_1,\dots,e_k > \oplus < e_{m+k+1},\dots,e_{2m} >$. Notice that W_o is a Lagrangian subspace of $\mathbb{C}^{2m}$, $< e_1,\dots,e_k >$ $(< e_{n+k+1},\dots,e_{2n} >)$ is a Lagrangian subspace of $\mathbb{C}^{2k}$ $(\mathbb{C}^{2(m-k)}$ $)$ respect to the symplectic form $\omega_k = X^tJ_kY$ $(\omega_{m-k}(X,Y) = X^tJ_{m-k}Y)$. Hence the orbit of K through W_o is $\mathrm{Sp}(k)/\mathrm{U}(k)\times\mathrm{Sp}(m-k)/\mathrm{U}(m-k)$, and the tangent space at $[\mathrm{U}(m)]$ splits

$$\mathrm{S}^2(\mathbb{C}^m) = \mathrm{S}^2(\mathbb{C}^k)\oplus\mathrm{S}^2(\mathbb{C}^{m-k})\oplus(\mathbb{C}^k\otimes\mathbb{C}^{m-k})^*,$$

as $\mathrm{U}(k)\times\mathrm{U}(m-k)-$modules, proving that the slice representation is given by $\mathbb{C}^k\otimes\mathbb{C}^{m-k}$ on which $\mathrm{U}(k)\otimes\mathrm{U}(m-k)$ acts. Note that the slice appears in Table Ia: this is another way to prove that the $K-$action is multiplicity free. Now let $L \subseteq K = \mathrm{Sp}(k)\times\mathrm{Sp}(m-k)$ and let $\mathfrak{l}$ be the Lie algebra of L. Suppose $\mathfrak{l}$ acts coisotropically. We consider the projections $\sigma_1 : \mathfrak{l} \longrightarrow \mathfrak{sp}(k)$, $\sigma_2 : \mathfrak{l} \longrightarrow \mathfrak{sp}(m-k)$ and we put $\mathfrak{l}_i = \sigma_i(\mathfrak{l})$. This means that $\mathfrak{l} \subset \mathfrak{l}_1+\mathfrak{l}_2$, $\mathfrak{l}_1+\mathfrak{l}_2$ acts coisotropically on $\mathrm{Sp}(m)/\mathrm{U}(m)$, so $\mathfrak{l}_1$, respectively $\mathfrak{l}_2$, acts coisotropically on $\mathrm{Sp}(k)/\mathrm{U}(k)$, respectively $\mathrm{Sp}(m-k)/\mathrm{U}(m-k)$. Then we have the following possibility

§1 $\mathfrak{l}_1$ and $\mathfrak{l}_2$ act both transitively

Hence $\mathfrak{l} = \mathfrak{sp}(k)+\mathfrak{sp}(m-k)$ or $\mathfrak{l} = \mathfrak{sp}(k)+\theta(\mathfrak{sp}(k))$, where θ is an automorphism of $\mathfrak{sp}(k)$. The first case corresponds to $\mathrm{Sp}(k)\times\mathrm{Sp}(m-k)$ that we have just considerated. The second case must be excluded by dimensional condition. Indeed, the dimension of a Borel subgroup of $\mathfrak{l}^{\mathbb{C}}$ is k^2+k while $\dim\mathrm{Sp}(2k)/\mathrm{U}(2k) = 2k^2+k$

§2 $\mathfrak{l}_1$ acts transitively and $\mathfrak{l}_2$ acts coisotropically

We must consider the following cases

(1) $\mathfrak{l}_1 = \mathfrak{sp}(k)$ and $\mathfrak{l}_2$ has a fixed point. Hence $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$. The orbit through W_o is a complex orbit and the slice is given by $\mathrm{S}^2(\mathbb{C}^{m-k}) \oplus (\mathbb{C}^{m-k} \oplus \mathbb{C}^k)^*$ on which $\mathfrak{u}(k)$ acts on $\mathbb{C}^k$ and $\mathfrak{l}_2$ acts on $\mathbb{C}^{m-k}$. By Tables IIa and IIb, this representations fails to be multiplicity free when $m-k \geq 2$. If $m-k=1$, note that $\mathfrak{l}_2$ must be $\mathfrak{u}(1)$, then the action is multiplicity free but the scalar cannot be removed. Summing up, we have the following multiplicity free action: $\mathfrak{l} = \mathfrak{sp}(m-1) + \mathfrak{u}(1)$

(2) $\mathfrak{l}_2 \subseteq \mathfrak{sp}(m_1) + \mathfrak{sp}(m_2)$, where $m_1 + m_2 = m - k$. We may suppose, up to conjugation in $\mathfrak{sp}(m)$, $k \geq m_1 \geq m_2$. Let $\mathfrak{l}_2 = \mathfrak{sp}(m_1) + \mathfrak{sp}(m_2)$. Then $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$, which corresponds to $L = \mathrm{Sp}(k) \times \mathrm{Sp}(m_1) \times \mathrm{Sp}(m_2) \subseteq K = \mathrm{Sp}(k) \times \mathrm{Sp}(m-k)$. We have proved that there exists $W \in \mathrm{Sp}(m-k)/\mathrm{U}(m-k)$ such that $\mathrm{Sp}(m_1) \times \mathrm{Sp}(m_2) W$ is a complex orbit. Since $\mathrm{Sp}(k) \times \mathrm{Sp}(m-k) W_o = \mathrm{Sp}(k)/\mathrm{U}(k) \times \mathrm{Sp}(m-k)/\mathrm{U}(m-k)$, the orbit $\mathrm{Sp}(k) \times \mathrm{Sp}(m_1) \times \mathrm{Sp}(m_2) W$ is a complex orbit and the slice is given by

$$(\mathbb{C}^k \otimes \mathbb{C}^{m_1})^* \oplus (\mathbb{C}^k \otimes \mathbb{C}^{m_2})^* \oplus (\mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2})^*$$

on which $\mathrm{U}(k)$ acts on $\mathbb{C}^k$, $\mathrm{U}(m_1)$ acts on $\mathbb{C}^{m_1}$ and $\mathrm{U}(m_2)$ acts on $\mathbb{C}^{m_2}$. By Tables IIa and IIb we must assume $m_1 = m_2 = 1$, so the slice becomes $(\mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C})^*$ and the two copies of $\mathrm{U}(1)$ act as $(e^{-i\psi}, 1, e^{-i\psi})$ and $(1, e^{-i\phi}, e^{-i\phi})$ respectively. Since a representation (ρ, V) is multiplicity free if and only if the dual representation (ρ^*, V^*) is, we may assume that $S = \mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}$. To solve this case we apply (ii) of Theorem 1.3. Note also by the Theorem 1.1 page 7 in [18] we may analyze the slice representation. Firstly, let $1 \in \mathbb{C}$. The orbit is S^1 and the slice is given by $\mathbb{R} \oplus \mathbb{C}^k \oplus \mathbb{C}^k$ on which $\mathrm{U}(1) \times \mathrm{U}(k)$ acts as follows: $(e^{i\phi}, A)(\alpha, v, w) = (\alpha, e^{i\phi} A v, e^{-i\phi} A w)$. Now, we consider $(0, 0, (1, \ldots, 0))$; the orbit is the unit sphere and the slice becomes $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}^{k-1}$ on which $\mathrm{T}^1 \times \mathrm{U}(k-1)$ acts as follow: $(e^{i\phi}, A)(\alpha, \beta, z, v) = (\alpha, \beta, e^{i\phi} z, A v)$. Now it is easy to see that $H_{\mathrm{princ}} = \mathrm{U}(k-2)$ and the cohomogeneity is 4, thus proving

$$4 = \mathrm{ch}(H, S) = \mathrm{rank}(H) - \mathrm{rank}(H_{\mathrm{princ}}) = 2 + k - (k-2).$$

We must analyze the behaviour of the subgroup of H. However, by the Restriction lemma [15], if one takes $L \subset \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ such that $\mathrm{Sp}(m-2) \times L$ acts coisotropically on $\mathrm{Sp}(m)/\mathrm{U}(m)$ then L acts coisotropically on $\mathrm{Sp}(2)/\mathrm{U}(2)$. Hence, for dimensional reasons, L must be $\mathrm{U}(1) \times \mathrm{Sp}(1)$. However, the orbit through W is a complex orbit and the slice becomes $(\mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}^k)^* \oplus (\mathbb{C})^* \oplus (\mathbb{C})^* \oplus \mathrm{S}^2(\mathbb{C})$ on which $\mathrm{U}(k)$ acts on $\mathbb{C}^k$, so by Tables IIa and IIb the action fails to be multiplicity free.

§3 $\mathfrak{l}_1$ and $\mathfrak{l}_2$ act both coisotropically

Since if both $\mathfrak{l}_1$ and $\mathfrak{l}_2$ have a fixed point, then $\mathfrak{l} \subseteq \mathfrak{l}_1 + \mathfrak{l}_2$ has a fixed point, we shall analyze the following cases: $\mathfrak{l}_1 = \mathfrak{u}(k)$, $\mathfrak{l}_2 = \mathfrak{sp}(m_1) + \mathfrak{sp}(m_2)$ and $\mathfrak{l}_1 = \mathfrak{sp}(k_1) + \mathfrak{sp}(k_2)$, $\mathfrak{l}_2 = \mathfrak{sp}(m_1) + \mathfrak{sp}(m_2)$. Since $\mathfrak{l}_1 + \mathfrak{l}_2 \subseteq \mathfrak{sp}(k) + \mathfrak{sp}(m_1) + \mathfrak{sp}(m_2)$, we have $m_1 = m_2 = 1$. In particular, the first case must be excluded for dimensional reason. In the second case $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$, which corresponds to $L = \mathrm{Sp}(k_1) \times \mathrm{Sp}(k_2) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ and one may prove that L has a complex orbit given by $\mathrm{Sp}(k_1)/\mathrm{U}(k_1) \times \mathrm{Sp}(k_2)/\mathrm{U}(k_2) \times \mathrm{Sp}(1)/\mathrm{U}(1) \times \mathrm{Sp}(1)/\mathrm{U}(1)$ whose slice representation fails to be multiplicity free.

4. $M = \mathrm{SO}(2m)/\mathrm{U}(m)$

In the following subsections we will go through all maximal subgroups K of $\mathrm{SO}(2m)$ according to Table **IV** in the Appendix.

4.1. **The case $K = \rho(H)$, H simple such that $\rho \in Irr_{\mathbb{R}}(H)$, $d(\rho) = 2m$.** By lemma 2.1 we shall analyze the cases $(\mathfrak{so}(6), \Lambda_3)$ and $(\mathfrak{so}(6), \Lambda_2)$, which correspond to the transitive action of $\mathrm{SO}(6)$ on $\mathrm{SO}(6)/\mathrm{U}(4)$.

4.2. **The fixed point case $K = \mathrm{U}(m)$.** We use the same notation and the same strategy as in section 3.2 By Table Ia $\mathrm{U}(m)$ acts coisotropically on $\Lambda^2(\mathbb{C}^m)$ and the scalar can be reduced. Throughout this section we denote by $\mathfrak{z}$ the center of Lie($\mathrm{U}(m)$)= $\mathfrak{u}(m)$ and by $\mathfrak{t}_m$ the Lie algebra of a maximal torus of $\mathfrak{su}(m) \subseteq \mathfrak{u}(m)$. Let L be a compact subgroup of $\mathrm{U}(m)$ such that $\mathfrak{l} = \mathfrak{z} + \mathfrak{l}_1$, where $\mathfrak{l}_1$ is a maximal Lie algebra of $\mathfrak{su}(m)$ (see Table **V** in the Appendix). By lemma 2.1 the case $\mathfrak{l}_1 = \mathfrak{so}(m)$ can be excluded, while the case $\mathfrak{l}_1 = \mathfrak{sp}(n), 2n = m$, appears when $n = 2$ and the slice becomes $\mathbb{C} \oplus \mathbb{C}^5$ on which $\mathrm{Sp}(2)/\mathbb{Z}_2 = \mathrm{SO}(5)$ acts on $\mathbb{C}^5$. Then $\mathfrak{l} = \mathfrak{z} + \mathfrak{sp}(2)$ acts coisotropically and the scalar cannot be removed. Notice that, since the slice of the orbit through $1 \in \mathbb{C}$ is $\mathbb{R} \oplus \mathbb{C}^5$ on which $\mathrm{SO}(5)$ acts on $\mathbb{C}^5$, one may prove, see also [13], the slice fails to be polar. This case is maximal, since for every $\mathfrak{h} \subseteq \mathfrak{sp}(2)$ we have $\mathfrak{z} + \mathfrak{h}$ does not satisfy the dimensional condition.

If $\mathfrak{l}_1 = \mathbb{R} + \mathfrak{su}(k) + \mathfrak{su}(m-k)$ then the slice becomes $\Lambda^2(\mathbb{C}^m) = \Lambda^2(\mathbb{C}^{m-k}) \oplus (\mathbb{C}^k \oplus \mathbb{C}^{m-k})^* \oplus \Lambda^2(\mathbb{C}^{m-k})$, on which $\mathfrak{su}(k)$, respectively $\mathfrak{su}(m-k)$, acts on $\mathbb{C}^k$, respectively $\mathbb{C}^{m-k}$. Hence by Tables Ia, Ib and Tables IIa, IIb, we have $k = 1$ and the slice becomes $\Lambda^2(\mathbb{C}^{m-1}) \oplus (\mathbb{C} \otimes \mathbb{C}^{m-1})^*$. The scalars, $\mathfrak{z}$ and $\mathbb{R} = \mathfrak{a}$, the centralizer of $\mathfrak{su}(m-1)$ in $\mathfrak{su}(m) \subseteq \mathfrak{u}(m)$, act as follows: let $(\psi, \theta) \in \mathfrak{a} \times \mathfrak{z}$, then $(\psi, \theta)(v, w) = (e^{2i(\theta - \frac{1}{m-1}\psi)}v, e^{-i(2\theta + \frac{m-2}{m-1}\psi)}w)$. Hence, the action is multiplicity free and we shall show how many centers we need. Firstly, we assume $m \geq 5$. By Table IIa the scalars can be reduced in the following cases: when $m-1$ is even, we need only a one dimensional center acting on the first submodule, that is satisfied with the line $\mathbb{R}(\frac{1}{m-1})$, where $\mathbb{R}(\alpha)$ means every line in the plane $(x, y) \in \mathfrak{a} \times \mathfrak{z}$ different from $y = \alpha x$, while when $m - 1 = 2s + 1$ one may prove that we can reduce the scalars, but the scalars cannot be removed. When $m = 4$, the slice becomes $(\mathbb{C}^3 \oplus \mathbb{C}^3)^*$, so by Table IIa, the scalars cannot be removed, but can be reduced if the center acts as (z^a, z^b) with $a \neq b$. This corresponds to $\mathbb{R}(0) + \mathfrak{su}(3)$. Finally, when $m = 3$, the slice becomes $\mathbb{C} \oplus \mathbb{C}^2$ and it is easy to see that the minimal subalgebra is $\mathfrak{z} + \mathfrak{su}(2)$. Notice that for $m \geq 4$ these actions are maximal by Tables IIa and IIb. If $m = 3$, then also $\mathfrak{z} + \mathfrak{t}_3$ acts coisotropically on $\mathrm{SO}(6)/\mathrm{U}(3)$ and when $m = 2$ we have also $\mathbb{R}(0)$, line in $\mathfrak{a} \times \mathfrak{z}$, acting on $\mathrm{SO}(4)/\mathrm{U}(2)$.

The case (iv) can be excluded by dimensional condition as in section 3.2. Indeed, $\dim_{\mathbb{C}} \mathrm{SO}(2m)/\mathrm{U}(m) = \frac{1}{2}pq(pq-1)$, since $m = pq$ and the dimension of a Borel subgroup of $(\mathrm{SU}(p) \otimes \mathrm{SU}(q))^{\mathbb{C}}$ is $\frac{1}{2}(p^2 + q^2 + p + q - 4)$. We shall prove that $pq(pq-1) > p^2 + q^2 + q + p - 2$ which implies that the dimensional condition is not satisfied for a Lie group with Lie algebra $\mathfrak{z} + \mathfrak{su}(p) + \mathfrak{su}(q)$.

Let $f(x) = x^2(q^2-1) - x(q+1) - q^2 - q + 2$. Then $f'(x) = 2x(q^2-1) - q + 4 > 0$, for $x \geq 3$ and $f(3) = 9(q^2-1) - 3(q+1) - q^2 - q + 4 > 0$, when $q \geq 2$.

Finally, we consider the case (v). By lemma 2.1 we have only the case $\mathfrak{su}(m)$ which has just been analyzed. Summing up, if $L \subset \mathrm{U}(m)$ acts coisotropically on M then, up to conjugation in $\mathfrak{o}(2m)$, the minimal algebra are in the following table

$\mathfrak{l}$	M	conditions
$\mathbb{R}(0)$	$\mathrm{SO}(4)/\mathrm{U}(2)$	$\mathbb{R}(0)$ line in $\mathfrak{t}_2 \times z$
$\mathfrak{z}+\mathfrak{t}_3$	$\mathrm{SO}(6)/\mathrm{U}(3)$	
$\mathbb{R}(\frac{1}{2k})+\mathfrak{su}(2k)$	$\mathrm{SO}(4k+2)/\mathrm{U}(2k+1)$	$k \geq 2,\ \mathbb{R}(\frac{1}{2k})$ line in $\mathfrak{a} \times \mathfrak{z}$
$\mathbb{R}+\mathfrak{su}(2k+1)$	$\mathrm{SO}(4k+4)/\mathrm{U}(2k+2)$	$k \geq 2$, $\mathbb{R}$ means any line in $\mathfrak{a} \times \mathfrak{z}$
$\mathbb{R}(0)+\mathfrak{su}(3)$	$\mathrm{SO}(8)/\mathrm{U}(4)$	$\mathbb{R}(0)$ line in $\mathfrak{a} \times \mathfrak{z}$
$\mathfrak{z}+\mathfrak{su}(2)$	$\mathrm{SO}(6)/\mathrm{U}(3)$	
$\mathfrak{su}(m)$	$\mathrm{SO}(m)/\mathrm{U}(m)$	$m \geq 2$
$\mathfrak{z}+\mathfrak{sp}(2)$	$\mathrm{SO}(8)/\mathrm{U}(4)$	

4.3. **The case** $K=\mathrm{SO}(p)\otimes\mathrm{SO}(q),\ 3\leq p\leq q$**.** By a straitforward calculation one may prove that $\mathrm{SO}(p)\otimes\mathrm{SO}(q),\ 3\leq p\leq q$ does not satisfy the dimensional condition.

4.4. **The case** $K=\mathrm{Sp}(p)\otimes\mathrm{Sp}(q),\ 4pq=2m$**.** One may prove that K does not satisfy the dimensional condition unless $p=q=1$ and $p=1$ and $q=2$. Now, the case $\mathrm{Sp}(1)\otimes\mathrm{Sp}(1)$ corresponds to the transitive action of $\mathrm{SO}(4)$ on $\mathrm{SO}(4)/\mathrm{U}(2)$, while $\mathrm{Sp}(1)\otimes\mathrm{Sp}(2)$ acts on $\mathrm{SO}(8)/\mathrm{U}(4)$. Since $\mathrm{Sp}(1)\otimes\mathrm{Sp}(2)\cap\mathrm{U}(4)=\mathrm{T}^1\cdot\mathrm{Sp}(2)$ the $\mathrm{Sp}(1)\otimes\mathrm{Sp}(2)-$orbit through $[\mathrm{U}(4)]$ is a complex orbit and the slice is given by $\mathbb{C}^5$, on which $\mathrm{Sp}(2)$ acts on $\mathbb{C}^5$ as $\mathrm{Spin}(5)/\mathbb{Z}_2=\mathrm{SO}(5)$. By Table Ia the action is multiplicity free and the scalar cannot be removed. Thanks to dimensional condition we must analyze only the following subgroups of $\mathrm{Sp}(1)\otimes\mathrm{Sp}(2)$: $H=\mathrm{T}^1\times\mathrm{Sp}(2)$, which has been considerated in the fixed point case, and $H=\mathrm{Sp}(1)\otimes(\mathrm{Sp}(1)\times\mathrm{Sp}(1))$. However, $H\cap\mathrm{U}(4)=\mathrm{T}^1\cdot(\mathrm{Sp}(1)\times\mathrm{Sp}(1))$ and the slice becomes $\Lambda^2(\mathbb{C}^2)\oplus(\mathbb{C}^2\otimes\mathbb{C}^2)^*\oplus\Lambda^2(\mathbb{C}^2)$ on which $\mathrm{Sp}(1)\otimes\mathrm{Sp}(1)$ acts on $\mathbb{C}^2\otimes\mathbb{C}^2$. By Table IIb, we need two dimensional scalars acting on $\mathbb{C}^2\otimes\mathbb{C}^2$, hence the action fails to be multiplicity free.

4.5. **The case** $K=\mathrm{SO}(k)\times\mathrm{SO}(2m-k)$**.** Since K is a symmetric group of $\mathrm{SO}(2m)$, the K-action is hyperpolar. We shall analyze the behaviour of the closed subgroups of $K=\mathrm{SO}(k)\times\mathrm{SO}(2m-k)$, so it is very useful to get a complex orbit of K. Notice that we may assume $k\leq m$. Firstly, we suppose $k=2s$. The homogeneous space $M=\mathrm{SO}(2m)/\mathrm{U}(m)$ parametrizes the almost complex structure $\mathbb{R}^{2m}$ that are orthogonal and compatible with a fixed orientation. Let J_1, respectively J_2, be almost complex structure of $\mathbb{R}^{2s}$, respectively $\mathbb{R}^{2(m-s)}$, as above and let $\mathbb{J}_o=J_1\oplus J_2$. Clearly, $\mathbb{J}_o$ is an orthogonal almost complex structure of $\mathbb{R}^{2m}$, the orbit $K\mathbb{J}_o$ is $\mathrm{SO}(2s)/\mathrm{U}(s)\times\mathrm{SO}(2m-2s)/\mathrm{U}(m-s)$ and the slice is given by $\mathbb{C}^s\otimes\mathbb{C}^{m-s}$ on which $\mathrm{U}(s)$ acts on $\mathbb{C}^s$ and $\mathrm{U}(m-s)$ acts on $\mathbb{C}^{m-s}$, i.e. $K\mathbb{J}_o$ is a complex orbit. If $k=2s+1$ we split $\mathbb{R}^{2n}=\mathbb{R}^{2s}\oplus\mathbb{R}^2\oplus\mathbb{R}^{2(m-s-1)}$ and we consider $\mathbb{J}_e=J_1\oplus J_2\oplus J_3$, where J_1, J_2 and J_3 are orthogonal almost complex structures of $\mathbb{R}^{2s}$, $\mathbb{R}^2$ and $\mathbb{R}^{2(m-s-1)}$ respectively. One may prove that the orbit through $\mathbb{J}_e$ is $\mathrm{SO}(2s+1)/\mathrm{U}(s)\times\mathrm{SO}(2(m-s-1)+1)/\mathrm{U}(m-s-1)$, and the slice is given by $(\mathbb{C}^s\otimes\mathbb{C}^{m-s-1})^*$.

Now let $L\subseteq K=\mathrm{SO}(2s)\times\mathrm{SO}(2m-2s)$ and let $\mathfrak{l}$ be the Lie algebra of L. Suppose $\mathfrak{l}$ acts coisotropically. We consider the projections $\sigma_1:\mathfrak{l}\longrightarrow\mathfrak{so}(k)$, $\sigma_2:\mathfrak{l}\longrightarrow\mathfrak{so}(2m-k)$ and we put $\mathfrak{l}_i=\sigma_i(\mathfrak{l})$. This means that $\mathfrak{l}\subset\mathfrak{l}_1+\mathfrak{l}_2$, $\mathfrak{l}_1+\mathfrak{l}_2$, acts coisotropically on $\mathrm{SO}(m)/\mathrm{U}(m)$, so $\mathfrak{l}_1$, respectively $\mathfrak{l}_2$, acts coisotropically on $\mathrm{SO}(2s)/\mathrm{U}(s)$, respectively on $\mathrm{SO}(2m-2s)/\mathrm{U}(m-s)$. In the sequel we refer to Tables Ia, Ib and Tables IIa, IIb in the Appendix, for all the conditions under which one can remove or reduce the scalar preserving the multiplicity free action. Then we

have the following possibility

§1 $\mathfrak{l}_1$ and $\mathfrak{l}_2$ act both transitively

For dimensional reasons $\mathfrak{l} = \mathfrak{so}(2s) + \mathfrak{so}(2(m-s))$ which has just been considerated.

§2 $\mathfrak{l}_1$ acts transitively and $\mathfrak{l}_2$ acts coisotropically

We must analyze the following cases

(1) $\mathfrak{l}_1 = \mathfrak{s}o(2m-2)$ and $\mathfrak{l}_2 = 0 \subseteq \mathfrak{so}(2)$. The orbit through $\mathbb{J}_o$ is complex and the slice becomes $(\mathbb{C} \otimes \mathbb{C}^{m-1})^*$ where $\mathfrak{u}(m-1)$ acts on $\mathbb{C}^{m-1}$. Hence, by Table Ia, the action is multiplicity free. Since the cohomogeneity is 1 this action is hyperpolar.

(2) $\mathfrak{l}_2$ has a fixed point. The orbit through $\mathbb{J}_o$ is a complex orbit $\mathrm{SO}(2s)/\mathrm{U}(s)$, so we are going to analyze the slice representation according the table appears in section 4.2.

- $\mathfrak{l}_1 = \mathbb{R}(0) \subseteq \mathfrak{u}(2) \subseteq \mathfrak{so}(4)$. The slice becomes
$$(\mathbb{C}^s \otimes \mathbb{C})^* \oplus (\mathbb{C}^s \otimes \mathbb{C})^* \oplus (\mathbb{C} \otimes \mathbb{C})^*.$$
on which $\mathfrak{u}(s)$ acts on $\mathbb{C}^s$ and $\mathbb{R}(0)$ acts on $\mathbb{C}$. Hence the action fails to be multiplicity free since the scalars act on $(\mathbb{C}^s \otimes \mathbb{C})^* \oplus (\mathbb{C}^s \otimes \mathbb{C})^*$ as a one dimensional scalar;
- the cases $\mathfrak{l}_2 = \mathfrak{z} + \mathfrak{t}_3$, $\mathfrak{l}_2 = \mathbb{R}(\frac{1}{2k}) + \mathfrak{su}(2k)$, $\mathfrak{l}_2 = \mathbb{R} + \mathfrak{su}(2k+1)$, $k \geq 2$. $\mathfrak{l}_2 = \mathbb{R}(0) + \mathfrak{su}(3)$ and $\mathfrak{l}_2 = \mathfrak{z} + \mathfrak{sp}(2)$ can be excluded since two many terms appear in the slice. Indeed, for example, let $\mathfrak{l}_2 = \mathbb{R}(\frac{1}{2k}) + \mathfrak{su}(2k)$. Then $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$, and the slice becomes $(\mathbb{C}^s \oplus \mathbb{C}^{2k})^* \oplus (\mathbb{C}^s \oplus \mathbb{C})^* \oplus \Lambda^2(\mathbb{C}^{2k}) \oplus (\mathbb{C} \oplus \mathbb{C}^{2k})^*$. By Tables IIa and IIb this action is not multiplicity free.
- $\mathfrak{l}_2 \subset \mathfrak{z} + \mathfrak{su}(m-s)$. The slice becomes $(\mathbb{C}^s \otimes \mathbb{C}^{m-s})^* \oplus \Lambda^2(\mathbb{C}^{m-s})$ on which $\mathfrak{u}(s)$ acts on $\mathbb{C}^s$ and $\mathfrak{l}_2$ acts on $\mathbb{C}^{m-s}$. If $m-s \geq 4$ then the action fails to be multiplicity free while if $m-s = 3$ or $m-s = 2$ then the action is multiplicity free with the scalar $\mathfrak{z}$. Summing up we have the following subalgebra: $\mathfrak{so}(2m-6) + \mathfrak{u}(3)$, $m \geq 5$, and $\mathfrak{so}(2m-4) + \mathfrak{u}(2)$, $m \geq 4$ acting on $\mathrm{SO}(2m)/\mathrm{U}(m)$.

(3) $\mathfrak{l}_2 = \mathfrak{sp}(1) + \mathfrak{sp}(2)$. Then $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$ and a complex orbit is given by $\mathrm{SO}(2(m-4)/\mathrm{U}(m)) \times \mathbb{C}$. However, one may prove that the slice fails to be multiplicity free;

(4) $\mathfrak{l}_2 \subseteq \mathfrak{so}(m_1) + \mathfrak{so}(m_2)$. We may assume, up to conjugation, that $2s \geq m_1 \geq m_2$. Let $\mathfrak{l}_2 = \mathfrak{so}(m_1) + \mathfrak{so}(m_2)$. Then $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$ which corresponds to $\mathrm{SO}(2s) \times \mathrm{SO}(m_1) \times \mathrm{SO}(m_2)$. Assume both m_1 and m_2 are even. We know that there exists $\mathbb{J}_o$ such that $\mathrm{SO}(m_1) \times \mathrm{SO}(m_2)\mathbb{J}_o$ is a complex orbit in $\mathrm{SO}(2m-2s)/\mathrm{U}(m-s)$. Hence $\mathrm{SO}(2s) \times \mathrm{SO}(m_1) \times \mathrm{SO}(m_2)\mathbb{J}_o$ is a complex orbit and the slice is given by
$$(\mathbb{C}^s \otimes \mathbb{C}^{\frac{m_2-1}{2}})^* \oplus (\mathbb{C}^s \otimes \mathbb{C}^{\frac{m_1-1}{2}})^* \oplus (\mathbb{C}^{\frac{m_1-1}{2}} \otimes \mathbb{C}^{\frac{m_2-1}{2}})^*.$$
Since $s \geq 2$, by Tables IIa and IIb we get $m_1 = m_2 = 2$ and the slice becomes
$$(\mathbb{C}^s \otimes \mathbb{C})^* \oplus (\mathbb{C}^s \otimes \mathbb{C})^* \oplus (\mathbb{C} \otimes \mathbb{C})^*$$
on which $\mathrm{U}(s)$ acts on $\mathbb{C}^s$. The center of $\mathrm{U}(s)$ acts as as $(e^{-i\theta}, e^{-i\theta}, 1)$, while $\mathrm{SO}(2) \times \mathrm{SO}(2)$ acts as $(e^{-i\phi}, e^{-i\psi}, e^{-i(\phi+\psi)})$. Hence, we get the following minimal subalgebra: $\mathfrak{so}(4) + \mathbb{R} + \mathbb{R}$ acting on $\mathrm{SO}(8)/\mathrm{U}(4)$ and $\mathfrak{so}(2s) + \mathbb{R}(1,-1)$, where $\mathbb{R}(1,-1)$ is a line different form $y = x, y = -x$, acting on $\mathrm{SO}(2(s+2))/\mathrm{U}(s+2)$, for $s \geq 3$.

Finally, assume that m_1 and m_2 are odd. Notice that the case $m_1 = m_2 = 1$ has been considerated. Hence the slice of the complex orbit $\mathrm{SO}(2s)\times\mathrm{SO}(m_1)\times\mathrm{SO}(m_2)\mathbb{J}_e$ is given by $(\mathbb{C}^s\otimes\mathbb{C}^{\frac{m_1-1}{2}})^*\oplus(\mathbb{C}^s\otimes\mathbb{C}^{\frac{m_2-1}{2}})^*\oplus(\mathbb{C}^s\oplus\mathbb{C})^*\oplus(\mathbb{C}^{\frac{m_1-1}{2}}\otimes\mathbb{C}^{\frac{m_2-1}{2}})^*$ so this action is not multiplicity free.

§3 $\mathfrak{l}_1$ and $\mathfrak{l}_2$ act both coisotropically

As in section 3.4 we may prove that $\mathfrak{l}$ does not act coisotropically. For example, let $\mathfrak{l}_1 = \mathfrak{u}(l)$ and let $\mathfrak{l}_2 = \mathfrak{so}(p) + \mathfrak{so}(q)$, where p, q are even. Then $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$ and the orbit through through $\mathbb{J}_o'\oplus\mathbb{J}_o$ is a complex orbit whose slice is given by $\Lambda^2(\mathbb{C}^l)\oplus(\mathbb{C}^{\frac{p}{2}}\otimes\mathbb{C}^{\frac{q}{2}})^*\oplus(\mathbb{C}^l\otimes\mathbb{C}^{\frac{p}{2}})^*\oplus,(\mathbb{C}^l\otimes\mathbb{C}^{\frac{q}{2}})^*$, on which $\mathfrak{u}(l)$ acts on $\mathbb{C}^l$, and $\mathfrak{u}(\frac{p}{2})$, respectively $\mathfrak{u}(\frac{q}{2})$, acts on $\mathbb{C}^{\frac{p}{2}}$, respectively $\mathbb{C}^{\frac{q}{2}}$. Hence, this action fails to be multiplicity free.

Now we are going to analyze the behaviour of the subgroup of $\mathrm{SO}(k)\times\mathrm{SO}(2m-k)$ when k is odd. The maximal subgroup L of $\mathrm{SO}(k)\times\mathrm{SO}(2m-k)$ are: $H\times\mathrm{SO}(2m-k)$, where H is a maximal subgroup of $\mathrm{SO}(k)$, $\mathrm{SO}(k)\times H$ where H is a maximal subgroup of $\mathrm{SO}(2m-k)$ and when $k = 2m-k$, $\mathrm{SO}(k)\times A(\mathrm{SO}(k))$ where A is an automorphism of $\mathrm{SO}(k)$. However, the last case can be excluded for dimensional reasons.

Since k is even we have the following cases: $H = \mathrm{SO}(p)\otimes\mathrm{SO}(q)$, $pq = k$, $3\leq p\leq q$ and $H = \sigma(L)$, L simple such that $\sigma\in Irr_{\mathbb{R}}(L)$. The first case may excluded by dimensional reasons. Indeed, if $H\times\mathrm{SO}(2m-k)$ acts coisotropically on $M = \mathrm{SO}(2m)/\mathrm{U}(m)$ then, by Restriction lemma, see [15], $H\times\mathrm{SO}(2m-k)$ acts coisotropically on the complex orbit of $\mathrm{SO}(k)\times\mathrm{SO}(2m-k)$, that is $\mathrm{SO}(2s+1)/\mathrm{U}(s)\times\mathrm{SO}(2(m-s-1)+1)/\mathrm{U}(m-s-1)$, since $k = 2s+1$. In particular H acts coisotropically on $\mathrm{SO}(2s+1)/\mathrm{U}(s)$. However the dimension of a Borel subgroup of $H^{\mathbb{C}}$ is lesser than $\frac{p^2+q^2}{4}$ while $\dim\mathrm{SO}(2s+1)/\mathrm{U}(s) = \frac{p^2q^2-1}{8}$, since $s = \frac{pq-1}{2}$. The inequality $2(p^2+q^2) < p^2q^2-1$ means that the dimensional condition does not satisfy.

Let $f(x) = x^2(p^2-2)-2p^2-1$. Then $f'(x) > 0$ if $x > 0$ and $f(3) = p^2-19 > 0$. Hence the action fails to be multiplicity free.

Now, we shall prove that if $H = \sigma(L)$, L simple such that $\sigma\in Irr_{\mathbb{R}}(L)$ then $H = G_2\subseteq\mathrm{SO}(7)$. As before, if $H\times\mathrm{SO}(2m-k)$ acts coisotropically then the dimension of a Borel subalgebra of $\mathfrak{h}$ must satisfy the following inequality

$$\dim\mathfrak{b}\geq\frac{d^2-1}{8} \tag{4.1}$$

We may analyze any simple Lie algebra as in lemma 2.1. Notice that $d(\sigma)$ must be odd. This is a straitforward calculation and easy to check. We demonstrate our method analyzing the cases $\mathfrak{h} = \mathfrak{su}(m)$ and $\mathfrak{g}_2$.

If $\mathfrak{h} = \mathfrak{su}(m)$, then $\dim\mathfrak{b} = \frac{1}{2}(m-1)(m+2)$. The case $m = 2$ give rise a real representation $2\Lambda_1$ which corresponds to the transitive action of $\mathrm{SO}(3)$. Now, assume $m\geq 3$. It is well known that if $\sigma = \sum_{i=1}^{m-1} a_i\Lambda_i$ is a contragradient representation then $a_i = a_{m-i}$, and one may prove that $d(\sigma)\geq d(\Lambda_1+\Lambda_{m-1})$. Since $d(\Lambda_1+\Lambda_{m-1}) = m^2-1\geq\frac{5}{2}m$, (4.1) does not hold for any real representation. Assume $\mathfrak{h} = \mathfrak{g}_2$. Since the dimension of a Borel subalgebra is 8 hence (1) becomes $63\geq d^2(\sigma)$ that is verified only for Λ_1 which corresponds to $\mathrm{G}_2\subseteq\mathrm{SO}(7)$ acting on $M = \mathrm{SO}(8)/\mathrm{U}(4)$. Since $\mathrm{G}_2\cap\mathrm{U}(4) = \mathrm{SU}(3)$, the orbit through $[\mathrm{U}(4)]$, $\mathrm{G}_2/\mathrm{SU}(3)\cong S^6$, is totally real. Indeed, let $\phi : \mathrm{SO}(8)/\mathrm{U}(4)\longrightarrow\mathfrak{g}_2^*$ be the moment map. Then $\mathrm{G}_2\phi([\mathrm{U}(4)]) = \mathrm{G}_2/P$ is a flag manifold, and $\mathrm{SU}(3)\subseteq P$. However $\mathrm{SU}(3)$

is a maximal subgroup of G_2 so $P = G_2$ and $\phi([\mathrm{U}(4)]) = 0$. Now, it is easy to check that $\mathrm{G}_2[\mathrm{U}(4)]$ is totally real. Moreover, since $2\dim_{\mathbb{R}} \mathrm{G}_2/\mathrm{SU}(3) = \dim_{\mathbb{R}} \mathrm{SO}(8)/\mathrm{U}(4)$, the slice representation can be deduced immediately from the isotropic representation of $\mathrm{SU}(3)$ on $\mathrm{G}_2/\mathrm{SU}(3)$, showing that the cohomogenity of the G_2-action is 1, which implies G_2 acts hyperpolarly on $\mathrm{SO}(8)/\mathrm{U}(4)$.

Now shall investigate $\mathrm{G}_2 \times \mathrm{SO}(2s+1)$, for every $s \geq 1$, acting on $\mathrm{SO}(2(s+4))/\mathrm{U}(s+4)$. The isotropy group of $\mathrm{G}_2 \times \mathrm{SO}(2s+1)\mathbb{J}_o$, is $\mathrm{SU}(3) \times \mathrm{U}(s)$ and the slice, from real point of view, is given by $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^s)$ on which $\mathrm{SU}(3)$ acts on $\mathbb{C}^3$ and $\mathrm{U}(s)$ acts on $\mathbb{C}^s$. We shall prove that (ii) of Theorem 1.3 is not satisfied. By the slice theorem, see [18], it is enough to study the slice representation.

The case $s = 1$ is a straitforward calculation and for dimensional reasons we shall assume $s \geq 3$. Let $v \in \mathbb{C}^3$ and let $w \in \mathbb{C}^s$ be two unit vectors. One can prove that the isotropy group of $v+v\otimes w$ is $\mathrm{SU}(2)\times\mathrm{U}(s-1)$ which acts on the slice $\mathbb{C}^2\oplus\mathbb{C}^2\otimes\mathbb{C}^{s-1}$. If we iterate this procedure two times then we get that the regular isotropy is $\mathrm{U}(s-3)$ and the cohomogeneity is 7. However $7 \neq \mathrm{rank}(\mathrm{G}_2\times\mathrm{SO}(2s+1))-\mathrm{rank}(\mathrm{U}(s-3)) = 5$.

Finally, we shall analyze $\mathrm{G}_2 \times \mathrm{G}_2$, acting on $\mathrm{SO}(14)/\mathrm{U}(7)$. However, for dimensional reasons, the action fails to be multiplicity free.

5. $M = \mathrm{E}_7/\mathrm{T}^1 \cdot \mathrm{E}_6$

In this section we analyze the behaviour of the subgroup of E_7. By dimensional condition, a subgroup $K \subseteq \mathrm{E}_7$ which acts coisotropically on M must satisfy $\dim K \geq 47$. The maximal subgroups of E_7 which satisfy the above inequality (see [18] page 41) are the following

maximal rank	$\mathrm{T}^1 \cdot \mathrm{E}_6$	$\mathrm{SU}(2) \cdot \mathrm{Spin}(12)$	$\mathrm{SU}(8)/\mathbb{Z}_2$
no maximal rank	$\mathrm{SU}(2) \cdot \mathrm{F}_4$		

We are going to analyze these cases separately.

5.1. **The fixed point case $K = \mathrm{T}^1 \cdot \mathrm{E}_6$.** The subgroup K acts coisotropically, since it has a fixed point and the slice representation, which is given by $(\mathbb{C}^{27}, \Lambda_1)$, appears in Table Ia. Note also that the scalar cannot be removed. The unique maximal subgroup H of $\mathrm{T}^1 \cdot \mathrm{E}_6$ which satisfies $\dim H \geq 47$ is $\mathrm{T}^1 \cdot \mathrm{F}_4$. However this actions fails to be multiplicity free. Indeed, the slice representation is given by $\mathbb{C}^{26} \oplus \mathbb{C}$, (see [1] lemma 14.4 page 95) so by Table Ia this actions fails to be multiplicity free.

5.2. **The case $K = \mathrm{SU}(2) \cdot \mathrm{F}_4$.** By Table 25 in [9] page 204, one sees, after conjugation, F_4 is contained in E_6. Hence the connected component of $K\cap\mathrm{T}^1\cdot\mathrm{E}_6$ is F_4 or $\mathrm{T}^1\cdot\mathrm{F}_4$, since K is a maximal subgroup. However $\mathbb{C}^{27} = \mathbb{C}^{26}\oplus\mathbb{C}$ as F_4-submodules (see Lemma 14.4 page 95 [1]). Hence $K \cap \mathrm{T}^1 \cdot \mathrm{E}_6 = \mathrm{T}^1 \cdot \mathrm{F}_4$ so the orbit through $[\mathrm{T}^1 \cdot \mathrm{E}_6]$ is a complex orbit which slice representation fails to be multiplicity free.

5.3. **The case $K = \mathrm{SU}(2)\cdot\mathrm{Spin}(12)$.** K is a symmetric group of E_7 hence the action is hyperpolar on M. Now, since any automorphism of E_7 is an inner automorphism then for any $\sigma, \tau \in \mathrm{Aut}(\mathrm{E}_7)$ there exists an element $g \in \mathrm{E}_7$ such that σ and $Ad(g^{-1})\circ\tau\circ Ag(g)$ commute. Hence we may assume that $K\cap\mathrm{T}^1\cdot\mathrm{E}_6$ is a symmetric subgroup of K and $\mathrm{T}^1 \cdot \mathrm{E}_6$. Since the symmetric subgroup of E_6 are the following

$\mathrm{T}^1 \cdot \mathrm{Spin}(10)$	$\mathrm{T}^1 \cdot \mathrm{SU}(6)$	F_4	$\mathrm{Sp}(4)/\mathbb{Z}_2$

then $K \cap \mathrm{T}^1 \cdot \mathrm{E}_6 = \mathrm{T}^1 \cdot (\mathrm{T}^1 \cdot \mathrm{Spin}(10))$, where the first T^1 lies in SU(2), but it is different from the centralizer of E_6 in E_7, while the second is the centralizer of Spin(10) in Spin(12). The slice representation is given by $\mathbb{C}^{16}$ on which $\mathrm{T}^1 \cdot \mathrm{T}^1 \cdot \mathrm{Spin}(10)$ acts. Hence K acts coisotropically on M.

Now we analyze the behaviour of the subgroup of K.

Let $L = \mathrm{T}^1 \cdot \mathrm{Spin}(12)$, where $\mathrm{T}^1 \subseteq \mathrm{SU}(2)$. Then $\mathrm{T}^1 \cdot \mathrm{Spin}(12) \cap \mathrm{T}^1 \cdot \mathrm{E}_6 = \mathrm{T}^1 \cdot (\mathrm{T}^1 \cdot \mathrm{Spin}(10))$ and the slice becomes $\mathbb{C}^{16} \oplus \mathbb{C}$, on which $\mathrm{T}^1 \cdot (\mathrm{T}^1 \cdot \mathrm{Spin}(10))$ acts. Note that the first scalar acts on $\mathbb{C}$ while the centralizer of Spin(10) in Spin(12) does not. Hence, the action is multiplicity free, since the Spin(10)$-$action on $\mathbb{C}^{16}$ is multiplicity free.

The case $L = \mathrm{Spin}(12)$ must be excluded, since $L \cap \mathrm{T}^1 \cdot \mathrm{E}_6 = \mathrm{T}^1 \cdot \mathbb{C}^{16}$, where T^1 is the centralizer of Spin(10) in Spin(12) and the slices becomes $\mathbb{C} \oplus \mathbb{C}^{16}$. However, the action on $\mathbb{C}$ is trivial. Then L does not act coisotropically on M.

Since $\mathbb{C}^{27} = \mathbb{C}^{16} \oplus \mathbb{C}^{10} \oplus \mathbb{C}$ as Spin(10)$-$submodules, one may prove that $\mathrm{SU}(2) \cdot \mathrm{T}^1 \cdot \mathrm{Spin}(10)$ fails to be multiplicity free. In particular, see Table **IV**, the subgroups H of K satisfying $\dim H \geq 47$, that we have not analyzed yet, are

$$\mathrm{SU}(2) \cdot \mathrm{Spin}(11),\ \mathrm{T}^1 \cdot \mathrm{Spin}(11),\ \mathrm{Spin}(11),\ \rho(H)\ H \text{ simple},\ \rho \in \mathrm{Irr}_{\mathbb{R}}(H),\ \mathrm{d}(\rho) = 12.$$

Let $H = \mathrm{SU}(2) \cdot \mathrm{Spin}(11)$. Since $K \cap \mathrm{T}^1 \cdot \mathrm{E}_6 = \mathrm{Spin}(10)$ then $H \cap \mathrm{T}^1 \cdot \mathrm{E}_6 = \mathrm{T}^1 \cdot \mathrm{Spin}(10)$, so the orbit of H through $[\mathrm{T}^1 \cdot \mathrm{Spin}(10)]$ is given by $\mathrm{Spin}(11)/\mathrm{Spin}(10) \times \mathbb{C}$. Note that H preserves the orbit $K[\mathrm{T}^1 \cdot \mathrm{E}_6]$, so the slice is given by $\mathbb{R}^{10} \oplus \mathbb{C}^{16}$, on which Spin(10) acts diagonally. Let $v \in \mathbb{R}^{10}$ be a unit vector. The orbit is the unit sphere on $\mathbb{R}^{10}$ and the slice becomes $\mathbb{R} \oplus \mathbb{C}^{16}$ where $\mathrm{T}^1 \cdot \mathrm{Spin}(9)$ acts on $\mathbb{C}^{16}$. This is the spin representation, and taking a unit real vector w, the isotropy group is Spin(7) and the slice becomes $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}^7 \oplus \mathbb{R}^8$ where Spin(7) acts both on $\mathbb{R}^8$ and on $\mathbb{R}^7$. Since $\mathrm{Spin}(7)/\ \mathrm{G}_2 = S^7$ and $\mathrm{G}_2/\mathrm{SU}(3) = S^5$, the regular isotropy is SU(3) and the cohomogeneity is 4. So we have $4 = \mathrm{rank}(\mathrm{SU}(2) \cdot \mathrm{Spin}(11)) - \mathrm{rank}(\mathrm{SU}(3))$, i.e. the action is multiplicity free. Notice that the slice fails to be polar (see [4]). Similarly we may prove that both the $\mathrm{T}^1 \cdot \mathrm{Spin}(11)$$-$action and Spin(11)$-$action fail to be multiplicity free Finally, the last case can be excluded by a straitforward calculation as lemma 2.1.

5.4. **The case $K = \mathrm{SU}(8)/\mathbb{Z}_2$.** K is a symmetric group of E_7 so K acts coisotropically on M. We are going to analyze its subgroups. Since $\mathrm{K} \cap \mathrm{T}^1 \cdot \mathrm{E}_6$ is a symmetric group of K and of $\mathrm{T}^1 \cdot \mathrm{E}_6$, we easily prove that $\mathrm{K} \cap \mathrm{T}^1 \cdot \mathrm{E}_6 = \mathrm{T}^1 \cdot \mathrm{SU}(2) \cdot \mathrm{SU}(6)$ and the slice becomes $\Lambda^2(\mathbb{C}^6)$ where $\mathrm{T}^1 \cdot \mathrm{SU}(6)$ acts. Indeed, K is a symmetric group and the orbit through $[\mathrm{T}^1 \cdot \mathrm{E}_6]$ is a complex orbit so the slice must be a multiplicity-free representation with degree 15. By Tables Ia, Ib and Tables IIa, IIb we get that the unique possibility is $\Lambda^2(\mathbb{C}^6)$.

By Table **V** and dimensional reasons we may investigate only $\mathrm{S}(\mathrm{U}_1 \times \mathrm{U}_7)$, SU(7) and $\rho(H)$, H is a simple group, such that $\rho \in Irr_{\mathbb{C}}(H)$ with $d(\rho) = 8$. The last case can be excluded by a straitforward calculation, while $\mathrm{S}(\mathrm{U}_1 \times \mathrm{U}_7)$ acts multiplicity free. Indeed, the orbit of K through $[\mathrm{T}^1 \cdot \mathrm{E}_6]$ is a complex orbit, that is $\mathrm{SU}(8)/\mathrm{S}(\mathrm{U}_2 \times \mathrm{U}_6)$, the complex Grassmannians of two plane. We may consider the plane $\pi = \langle e_1, e_2 \rangle$ so the orbit $\mathrm{S}(\mathrm{U}_1 \times \mathrm{U}_7)\pi$ is the complex orbit $\mathrm{S}(\mathrm{U}_1 \times \mathrm{U}_7)/\mathrm{S}(\mathrm{U}_1 \times \mathrm{U}_1 \times \mathrm{U}_6)$ which slice in M is given by $\mathbb{C}^6 \oplus \Lambda^2(\mathbb{C}^6)$. By Table IIa this action is multiplicity free. Notice that the slice is not polar. Similarly, one may prove that also SU(7) acts coisotropically, but non-polarly, on $\mathrm{E}_7/\mathrm{T}^1 \cdot \mathrm{E}_6$.

6. $M = \mathrm{E}_6/\mathrm{T}^1 \cdot \mathrm{Spin}(10)$

In this section we analyze the behaviour of the subgroup of E_6. By dimensional condition, if a subgroup $K \subseteq \mathrm{E}_6$ acts coisotropically on $M = \mathrm{E}_6/\mathrm{T}^1 \cdot \mathrm{Spin}(10)$, then $\dim K \geq 26$. The maximal subgroups of E_6 which satisfy the above inequality (see [18] page 41) are the following

maximal rank	$\mathrm{T}^1 \cdot \mathrm{Spin}(10)$	$\mathrm{SU}(2) \cdot \mathrm{Spin}(12)$	$\mathrm{Sp}(1) \cdot \mathrm{SU}(6)$
no maximal rank	$\mathrm{Sp}(4)$	F_4	

6.1. **The fixed point case** $K = \mathrm{T}^1 \cdot \mathrm{Spin}(10)$**.** K acts coisotropically and the slice representation appears in Table Ia and the scalar can be removed. Now, by Table **IV**, we shall analyze the following cases.

(1) $H = \mathrm{T}^1 \cdot \mathrm{Spin}(k) \times \mathrm{Spin}(10-k)$. Since $\dim H \geq 26$ we must consider only the cases $\mathrm{T}^1 \cdot \mathrm{Spin}(9)$, $\mathrm{T}^1 \cdot (\mathrm{T}^1 \times \mathrm{Spin}(8))$ and $\mathrm{T}^1 \cdot \mathrm{Spin}(8)$. The first one acts coisotropically but the scalar cannot be removed. In the other cases, the slice becomes $\mathbb{C}^{16} = \mathbb{C}^8 \oplus \mathbb{C}^8$, on which $\mathrm{Spin}(8)$, so $\mathrm{T}^1 \cdot (\mathrm{T}^1 \times \mathrm{Spin}(8))$ acts coisotropically but the scalar cannot be reduced. Notice that in these cases the slice fail to be polar (see [4] and [13]).

(2) $H = \mathrm{T}^1 \cdot \mathrm{U}(5)$. It is well know that the isotropy group of $[v]$ in $\mathbb{P}(\mathbb{C}^{16})$, where v is the highest weight is $\mathrm{U}(5)$. Moreover, the center of $\mathrm{U}(5)$ acts as scalar while $\mathrm{SU}(5)$ acts trivially on v. Hence $\mathrm{Spin}(10)v = \mathrm{Spin}(10)/\mathrm{SU}(5)$ and the isotropy representation is given by $\mathbb{C}^5 \oplus \Lambda^2(\mathbb{C}^5) \oplus \mathbb{R}$. In particular $\mathbb{C}^{16} = \mathbb{C}^5 \oplus \Lambda^2(\mathbb{C}^5) \oplus \mathbb{C}$, as $\mathrm{U}(5)-$submodules so by Table IIa this actions is multiplicity free. Notice that the slice fails to be polar by Theorem 2 [4] and for dimensional reasons any proper subgroup does not act coisotropically.

(3) $H = \mathrm{T}^1 \cdot \rho(H')$, where $\rho \in Irr_{\mathbb{C}}(H')$, $d(\rho) = 10$. This case can be excluded by a straitforward calculation as in lemma 2.1.

6.2. **The case** $K = \mathrm{SU}(2) \cdot \mathrm{SU}(6)$**.** K acts multiplicity-free since it is a symmetric group of E_6. We recall that in E_6 two involutions σ, τ commuting up to conjugation, i.e. there exists $g \in \mathrm{E}_6$ such that σ commutes with $\mathrm{Ad}(g) \circ \tau \circ \mathrm{Ad}(g^{-1})$ (see [6]). In particular we may assume that $K \cap \mathrm{T}^1 \cdot \mathrm{Spin}(10)$ is a symmetric group both of K and of $\mathrm{T}^1 \cdot \mathrm{Spin}(10)$. Hence, looking by the extended Dynkin diagram of E_6, we have $Lie(K \cap \mathrm{T}^1 \cdot \mathrm{Spin}(10)) = \mathbb{R} + (\mathbb{R} + \mathfrak{su}(5)) \subseteq \mathfrak{sp}(1) + \mathfrak{su}(6)$. Hence the orbit through $[\mathrm{T}^1 \cdot \mathrm{Spin}(10)]$ is a complex orbit and the slice is given by $\Lambda^2(\mathbb{C}^5)$. Now, we must consider the maximal subgroup of K. The group $\mathrm{T}^1 \cdot \mathrm{SU}(6)$ acts coisotropically since the orbit through $[\mathrm{T}^1 \cdot \mathrm{Spin}(10)]$ is $\mathbb{P}(\mathbb{C}^5)$ and the slice becomes $\mathbb{C} \oplus \Lambda^2(\mathbb{C}^5))$ on which $\mathrm{T}^1 \times \mathrm{U}(5)$ acts. In particular $\mathrm{SU}(6)$ does not act coisotropically since on the slice appears $\mathbb{C}$ on which the action is trivial. By dimensional condition, one may investigate only the following cases: $\mathrm{T}^1 \times \mathrm{S}(\mathrm{U}_1 \times \mathrm{U}_5)$ and $\mathrm{T}^1 \times \rho(H)$, H simple, $\rho \in Irr_{\mathbb{C}}(H)$, with $d(\rho) = 6$. The second case can be excluded by a straitforward calculation. In the first case, one may note that the orbit through $[\mathrm{T}^1 \cdot \mathrm{Spin}(10)]$ is a complex orbit and the slice becomes $\Lambda^2(\mathbb{C}^5) \oplus \mathbb{C}^5$ where $\mathrm{U}(5)$ acts diagonally. Hence the slice is a multiplicity free representation which is not polar by Theorem 2 in [4].

6.3. **The case** $K = \mathrm{Sp}(4)$**.** K is a symmetric group so the $K-$action is multiplicity free. By dimensional condition, we shall investigate the cases $\rho(H)$, H simple, ρ an irreducible representation of quaternionic type with $d(\rho) = 8$. However, it is easy to check that this case can be excluded.

6.4. **The case $K = \mathrm{F}_4$.** Since K is a symmetric group the $K-$action is multiplicity free. Moreover the unique maximal subgroup H which satisfies $\dim H \geq 26$ is $\mathrm{Spin}(9) \subseteq \mathrm{Spin}(10)$ so we fall again in the fixed point case.

7. Polar actions

In this section we study which coisotropic actions are polar. It is well known [21] that if a K-action is polar on M then every slice representation of K is polar. Notice also that the reducible actions arising from Tables IIa and IIb are not polar; this can be easily deduced as an application of Theorem 2 (page 313) [4], while see [13] and [18], in the irreducible case we know that $\mathfrak{u}(m)$ on $\mathrm{Sp}(m)/\mathrm{U}(m)$, $\mathfrak{u}(m)$ and $\mathfrak{su}(m)$ when m is odd on $\mathrm{SO}(2m)/\mathrm{U}(m)$, $\mathrm{Spin}(10)$ and $\mathrm{T}^1 \cdot \mathrm{Spin}(10)$ on $\mathrm{E}_6/\mathrm{T}^1 \cdot \mathrm{Spin}(10)$, $\mathrm{T}^1 \cdot \mathrm{E}_6$ on $\mathrm{E}_7/\mathrm{T}^1 \cdot \mathrm{E}_6$ give rise to hyperpolar actions. Moreover, any symmetric group and cohomogeneity one actions are hyperpolar. Hence we may consider the following cases: $\mathfrak{z} + \mathfrak{t}_3$ and $\mathfrak{z} + \mathfrak{su}(2)$ acting on $\mathrm{SO}(6)/\mathrm{U}(3)$, $\mathfrak{z} + \mathfrak{sp}(2)$ $\mathfrak{sp}(1) \otimes \mathfrak{sp}(2)$ acting on $\mathrm{SO}(8)/\mathrm{U}(4)$, $\mathrm{T}^1 \cdot \mathrm{Spin}(12)$ on $\mathrm{E}_7/\mathrm{T}^1 \cdot \mathrm{E}_6$ and finally $\mathfrak{sp}(m-1) + \mathfrak{u}(1)$ acting on $\mathrm{Sp}(m)/\mathrm{U}(m)$, Firstly, we consider $\mathrm{T}^1 \cdot \mathrm{Spin}(12)$ on $\mathrm{E}_7/\mathrm{T}^1 \cdot \mathrm{E}_6$. We recall that T^1 is not the centralizer of E_6 in E_7. In section 5.1 we have determined a complex orbit an its slice is given by $\mathbb{C} \oplus \mathbb{C}^{16}$ on which $\mathrm{T}^1 \cdot (\mathrm{T}^1 \cdot \mathrm{Spin}(10))$ acts. Hence the cohomogeneity is 3. If the action were polar the slice would be a compact non-flat locally symmetric space. Hence the slice must be a quotient of S^3 and its the tangent space is given by $\mathbb{R} + m$, where m is a section corresponding to the case $\mathrm{SU}(2) \cdot \mathrm{Spin}(12)$, so $[m, m] = 0$, since this action is hyperpolar. This means that the slice has an isometric group of rank at least two, which is a contradiction.

The case $\mathfrak{sp}(1) \otimes \mathfrak{sp}(2)$ can be excluded similarly. Indeed, we have proved that a slice is given by $\mathbb{C}^5$ on which $\mathrm{T}^1 \cdot \mathrm{SO}(5)$ acts. If the action were polar the section m would be an abelian subspace of dimension 2, i.e. the action would be hyperpolar which is a contradiction, see [18].

The other cases can be excluded using the same idea. For example, let $\mathfrak{l} = \mathfrak{z} + \mathfrak{su}(2)$. We have proved that the slice $\Lambda^2(\mathbb{C}^3) = \Lambda^2(\mathbb{C}^2) \oplus (\mathbb{C} \otimes \mathbb{C}^2)^*$, so that the action has cohomogeneity 2. If the action were polar a section can be taken as direct sum of the section for the action of T^1 on $\mathbb{C}$ plus a section for the $\mathrm{T}^1 \cdot \mathrm{SU}(2)$ action on $\Lambda^2(\mathbb{C}^3)$. Let $\mathfrak{m} =< X, Y >$, where

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2+i \\ 0 & -2-i & 0 \end{pmatrix} \in \Lambda^2(\mathbb{C}^2), \quad Y = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in (\mathbb{C} \otimes \mathbb{C}^2)^*.$$

One may prove that $[[X, Y], X]$ does not belong to $\mathfrak{m}$. Hence, by Theorem 7.2 page 226 [12] on Lie triple system, the section $\Sigma = \exp(\mathfrak{m})$ is not totally geodesic, hence the action cannot be polar.

8. Appendix

Table I a: Lie algebras $\mathfrak{k}$ s.t. $\mathbb{R}+\mathfrak{k}$ gives rise to irreducible multiplicity free actions

$\mathfrak{su}(n)$	$n \geq 1$	$\mathfrak{so}(n)$	$n \geq 3$
$\mathfrak{sp}(n)$	$n \geq 2$	$S^2(\mathfrak{su}(n))$	$n \geq 2$
$\Lambda^2(\mathfrak{su}(n))$	$n \geq 4$	$\mathfrak{su}(n) \otimes \mathfrak{su}(m)$	$n, m \geq 2$
$\mathfrak{su}(2) \otimes \mathfrak{sp}(n)$	$n \geq 2$	$\mathfrak{su}(3) \otimes \mathfrak{sp}(n)$	$n \geq 2$
$\mathfrak{su}(n) \otimes \mathfrak{sp}(2)$	$n \geq 4$	$\mathfrak{spin}(7)$	
$\mathfrak{spin}(9)$		$\mathfrak{spin}(10)$	
$\mathfrak{g}_2$	$n \geq 1$	$\mathfrak{e}_6$	$n \geq 3$

Table I b: Irreducible coisotropic actions in which the scalars are removable

$\mathfrak{su}(n)$	$n \geq 2$	$\mathfrak{sp}(n)$	$n \geq 2$
$\Lambda^2(\mathfrak{su}(n))$	$n \geq 4$	$\mathfrak{su}(n) \otimes \mathfrak{su}(m)$	$n, m \geq 2,\ n \neq m$
$\mathfrak{spin}(10)$		$\mathfrak{su}(n) \otimes \mathfrak{sp}(2)$	$n \geq 5$

Table II a: Indecomposable coisotropic actions in which the scalars can be removed or reduced

$\mathfrak{su}(n)\ \oplus_{\mathfrak{su}(n)}\ \mathfrak{su}(n)$	$n \geq 3,\ a \neq b$
$\mathfrak{su}(n)^*\ \oplus_{\mathfrak{su}(n)}\ \mathfrak{su}(n)$	$n \geq 3\ a \neq -b$
$\mathfrak{su}(2m)\ \oplus_{\mathfrak{su}(2m)}\ \Lambda^2(\mathfrak{su}(2m))$	$m \geq 2,\ b \neq 0$
$\mathfrak{su}(2m+1)\ \oplus_{\mathfrak{su}(2m+1)}\ \Lambda^2(\mathfrak{su}(2m+1))$	$m \geq 2,\ a \neq -mb$
$\mathfrak{su}(2m)^*\ \oplus_{\mathfrak{su}(2m)}\ \Lambda^2(\mathfrak{su}(2m))$	$m \geq 2,\ b \neq 0$
$\mathfrak{su}(2m+1)^*\ \oplus_{\mathfrak{su}(2m+1)}\ \Lambda^2(\mathfrak{su}(2m+1))$	$m \geq 2,\ a \neq mb$
$\mathfrak{su}(n)\ \oplus_{\mathfrak{su}(n)}\ (\mathfrak{su}(n) \otimes \mathfrak{su}(m))$	$2 \leq n < m,\ a \neq 0$
$\mathfrak{su}(n)\ \oplus_{\mathfrak{su}(n)}\ (\mathfrak{su}(n) \otimes \mathfrak{su}(m))$	$m \geq 2,\ n \geq m+2, a \neq b$
$\mathfrak{su}(n)^*\ \oplus_{\mathfrak{su}(n)}\ (\mathfrak{su}(n) \otimes \mathfrak{su}(m))$	$2 \leq n < m, a \neq 0$
$\mathfrak{su}(n)^*\ \oplus_{\mathfrak{su}(n)}\ (\mathfrak{su}(n) \otimes \mathfrak{su}(m))$	$2 \geq m, n \geq m+2,\ a \neq b$
$(\mathfrak{su}(2) \otimes \mathfrak{su}(2))\ \oplus_{\mathfrak{su}(2)}\ (\mathfrak{su}(2) \otimes \mathfrak{su}(n))$	$n \geq 3,\ a \neq 0$
$(\mathfrak{su}(n) \otimes \mathfrak{su}(2))\ \oplus_{\mathfrak{su}(2)}\ (\mathfrak{su}(2) \otimes \mathfrak{sp}(m))$	$n \geq 3, m \geq 4, b \neq 0$

Table II b: Indecomposable coisotropic actions in which the scalars cannot be removed or reduced

$\mathfrak{su}(2)\ \oplus_{\mathfrak{su}(2)}\ \mathfrak{su}(2)$	
$\mathfrak{su}(n)^{(*)}\ \oplus_{\mathfrak{su}(n)^*}\ (\mathfrak{su}(n) \oplus \mathfrak{su}(n))$	$n \geq 2$
$(\mathfrak{su}(n+1)^{(*)}\ \oplus_{\mathfrak{su}(n+1)}\ (\mathfrak{su}(n+1) \otimes \mathfrak{su}(n))$	$n \geq 2$
$(\mathfrak{su}(2)\ \oplus_{\mathfrak{su}(2)}\ (\mathfrak{su}(2) \otimes \mathfrak{sp}(m))$	$m \geq 2$
$(\mathfrak{su}(2) \oplus \mathfrak{su}(2))\ \oplus_{\mathfrak{su}(2)}\ (\mathfrak{su}(2) \otimes \mathfrak{sp}(m))$	
$(\mathfrak{sp}(n) \oplus \mathfrak{su}(2))\ \oplus_{\mathfrak{su}(2)}\ (\mathfrak{su}(2) \otimes \mathfrak{sp}(m))$	$n, m \geq 2$
$\mathfrak{sp}(n)\ \oplus_{\mathfrak{sp}(n)}\ \mathfrak{sp}(n)$	$n \geq 2$
$\mathfrak{spin}(8)\ \oplus_{\mathfrak{spin}(8)}\ \mathfrak{so}(8)$	

In the previous Tables we use the notation of [2], as an example $\mathfrak{su}(n) \oplus_{\mathfrak{su}(n)} \mathfrak{su}(n)$ denotes the Lie algebra $\mathfrak{su}(n)$ acting on $\mathbb{C}^n \oplus \mathbb{C}^n$ via the direct sum of two copies of the natural representation.

Table III: Maximal subgroups of $\mathrm{Sp}(m)$

i)	$\mathrm{U}(m)$	
ii)	$\mathrm{Sp}(k)\times\mathrm{Sp}(m-k)$	$1\leq k\leq m-1$
iii)	$\mathrm{SO}(p)\otimes\mathrm{Sp}(q)$	$pq=m,\ p\geq 3,\ q\geq 1$
iv)	$\rho(H)$	H simple, $\rho\in\mathrm{Irr}_{\mathbb{H}}(\mathrm{H}),\ \mathrm{d}(\rho)=2\mathrm{m}$

Table IV: Maximal subgroups of $\mathrm{SO}(m)$

i)	$\mathrm{SO}(k)\times\mathrm{SO}(m-k)$	$1\leq k\leq m-1$
ii)	$\mathrm{SO}(p)\otimes\mathrm{SO}(q)$	$pq=m,\ 3\leq p\leq q$
iii)	$\mathrm{U}(k)$	$2k=m$
iv)	$\mathrm{Sp}(p)\otimes\mathrm{Sp}(q)$	$4pq=m$
v)	$\rho(H)$	H simple, $\rho\in\mathrm{Irr}_{\mathbb{R}}(\mathrm{H}),\ \mathrm{d}(\rho)=\mathrm{m}$

Table V: Maximal subgroups of $\mathrm{SU}(m)$

i)	$\mathrm{SO}(m)$	
ii)	$\mathrm{Sp}(n)$	$2n=m$
iii)	$\mathrm{S}(\mathrm{U}_k\times\mathrm{U}_{m-k})$	$1\leq k\leq m-1$
iv)	$\mathrm{SU}(p)\otimes\mathrm{SU}(q)$	$pq=m,\ p\geq 3,\ q\geq 2$
v)	$\rho(H)$	H simple, $\rho\in\mathrm{Irr}_{\mathbb{C}}(\mathrm{H}),\ \mathrm{d}(\rho)=\mathrm{m}$

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